Nuclear Physics B55 (1973) 125–131. North-Holland Publishing Company

CRICHTON'S PHASE-SHIFT AMBIGUITY*

D. ATKINSON, P.W. JOHNSON**, N. MEHTA and M. de ROO
Instituut voor Theoretische Natuurkunde, P.O. Box 800, Groningen, The Netherlands

Recieved 11 January 1973

Abstract: A re-examination of the SPD phase-shift ambiguity is made with a view to understanding certain singular features of the elastic unitarity constraint. An explicit solution of Crichton's equations is presented, and certain features of this solution are displayed graphically. In particular, it is shown that there are two critical values of the D-wave phase shift, for which the locally linearized unitarity equation has a bifurcation point, for which the modulus of the amplitude has a zero at a physical point, $-1 < \cos\theta < 0$, and for which the real part has in addition a zero at $\cos\theta = 1$.

In fixed-energy phase-shift analysis, one tries effectively to find a complex scattering amplitude that is consistent with unitarity, and which has a given modulus. The question is especially simple in the case of elastic scattering, below the inelastic threshold, since then unitarity is an equality constraint on the real phase shifts. It is known in fact that different sets of phase shifts may lead to amplitudes with the same modulus (and so they would correspond to the same differential cross section). In particular, Crichton [1] has displayed an ambiguity involving only S, P and D waves.

We propose in this paper to investigate the Crichton ambiguity more fully, in order to understand the way that solutions of the unitarity equation can be non-unique. We will stress those features of this particular example that are expected to be of importance in more general cases, in particular the existence of zeros of the real part of the amplitude in the physical region, and the occurrence of critical points, at which the two solutions coalesce, and at which the modulus of the amplitude (and so the cross section) has a zero in the physical region. These critical points, of which there are two for the Crichton ansatz, correspond to bifurcation points of the locally linearized form of the unitarity equation [2,3]. Since we know that this equation defines a compact mapping [3], and we observe that there are two Crichton solutions emanating from each critical point, we may expect the kernel of the linearized equation to have an eigenvalue at the critical points. These features of the solutions are of especial relevance to the general problem of solving

---

* Supported in part by a NATO grant.
** Permanent address: Illinois Institute of Technology, Chicago.

the linearized equation by means of the Newton–Kantorovich iteration. More definitely, if one wishes to vary some parameters (for example, the cross section itself, or, in the inelastic region, the inelastic term), then the above-mentioned zeros and bifurcation points give rise to constraints between the parameter variations, as has been noted before [2,3].

We may write the square of the modulus of the scattering amplitude as follows:

$$|F(z)|^2 = \sum_k \sum_l (2k+1)(2l+1) P_k(z) P_l(z) \sin\delta_k \sin\delta_l \cos(\delta_k - \delta_l)\,, \tag{1}$$

where the phase shifts δ_l are to be real. Crichton truncated the series at the D wave and looked for solutions of the equation

$$|F(z)|^2 = |F'(z)|^2\,. \tag{2}$$

By equating powers of z, and after some straightforward manipulation, Crichton found the following constraints between $\delta_0, \delta_1, \delta_2$ and $\delta_0', \delta_1', \delta_2'$:

$$\alpha_0 \overset{\text{def}}{=} \delta_0 + \delta_0' = \tan^{-1}\left\{\frac{\sin\delta_2 \cos\delta_2}{0.2 - \sin^2\delta_2}\right\}, \tag{3}$$

$$\alpha_1 \overset{\text{def}}{=} \delta_1 + \delta_1' = \tfrac{1}{2}\pi + \delta_2\,, \tag{4}$$

$$\sin^2\delta_0 + 3\sin^2\delta_1 = \sin^2\delta_0' + 3\sin^2\delta_1'\,, \tag{5}$$

$$\sin^2\delta_1 + \sin\delta_1 \sin(2\delta_0 - \delta_1) = \sin^2\delta_1' + \sin\delta_1' \sin(2\delta_0' - \delta_1')\,, \tag{6}$$

where one has taken $\delta_2' = +\delta_2$ to eliminate the trivial ambiguity ($\delta_l' = -\delta_l$ for all l) at the outset.

It is in fact possible to solve eqs. (5) and (6):

$$\epsilon_0 \overset{\text{def}}{=} \delta_0 - \delta_0' = \cos^{-1}\left\{\frac{\sin\alpha_1}{\sin(\alpha_0 - \alpha_1)} - \frac{9}{4}\frac{\sin\alpha_1 \sin(\alpha_0 - \alpha_1)}{\sin^2\alpha_0} + \frac{1}{4}\frac{\sin(\alpha_0 - \alpha_1)}{\sin\alpha_1}\right\}, \tag{7}$$

$$\epsilon_1 \overset{\text{def}}{=} \delta_1 - \delta_1' = \cos^{-1}\left\{\frac{1}{3}\frac{\sin\alpha_0}{\sin(\alpha_0 - \alpha_1)} + \frac{3}{4}\frac{\sin(\alpha_0 - \alpha_1)}{\sin\alpha_0} - \frac{1}{12}\frac{\sin\alpha_0 \sin(\alpha_0 - \alpha_1)}{\sin^2\alpha_1}\right\}. \tag{8}$$

Eqs. (3), (4), (7) and (8) give $\alpha_0, \alpha_1, \epsilon_1, \epsilon_2$, and hence $\delta_0, \delta_0', \delta_1, \delta_1'$ as explicit functions of δ_2, which we shall take to be a parameter. In order to pick the correct quadrant for the inverse cosines, we need simply to check the equality

$$\sin\alpha_0 \sin\epsilon_0 + 3\sin\alpha_1 \sin\epsilon_1 = 0\,. \tag{9}$$

The general features of the solutions can best be appreciated graphically. For δ_2 in the first quadrant, the quantities ϵ_0 and ϵ_1 are real only when $\delta_2^m \leqslant \delta_2 \leqslant \delta_2^M$,

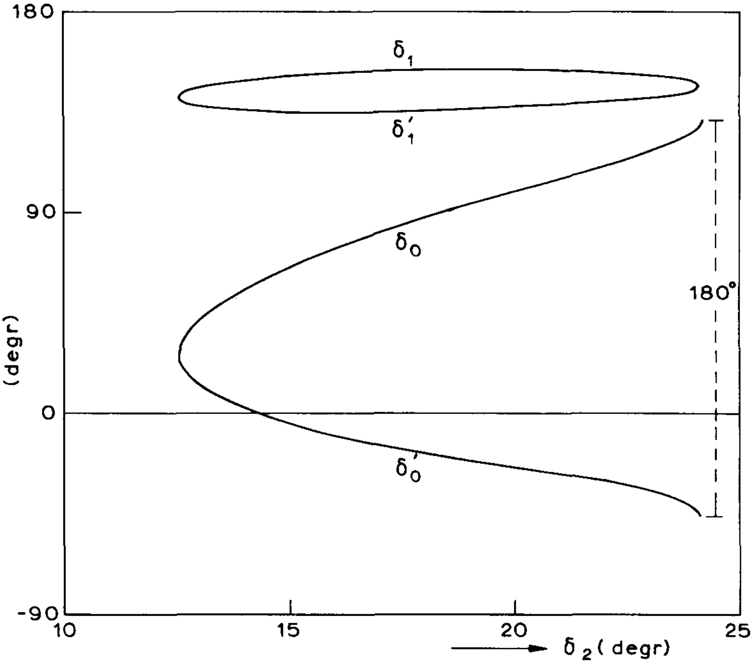

Fig. 1. The S- and P-wave phase shifts for the amplitudes F and F', plotted against the D-wave phase shift.

where $\delta_2^m \approx 12.53°$ and $\delta_2^M \approx 24.15°$. In fig. 1 we plot δ_0, δ_0', δ_1, and δ_1' against δ_2. Note that $\delta_0 = \delta_0'$, $\delta_1 = \delta_1'$ (*modulo* π), at the critical points δ_2^m and δ_2^M. We see that there is a unique set of phase shifts at $\delta_2 = \delta_2^m$, but that this set splits into two as δ_2 is increased. At $\delta_2 = \delta_2^M$, the phase shifts coincidence again (*modulo* π); and, outside the interval (δ_2^m, δ_2^M), there are no real solutions of eq. (2). In fig. 2, we show the real part of the amplitude, Re F, as a function of z, for $\delta_2 = \delta_2^m$ and $\delta_2 = \delta_2^M$ (solid curves), and for the representative intermediate value of 19° (dotted curves), for which there are of course two solutions, ReF and ReF'. It will be observed that, for both of the critical points δ_2^m and δ_2^M, there is a zero of ReF at $z = 1$, and another one at a negative, physical value of z. For intermediate values of δ_2, ReF has two physical zeros, one positive and one negative, while ReF' has only one physical zero, which is at a negative point (it actually has also a real, unphysical zero at some point $z > 1$). When δ_2 tends to δ_2^m or δ_2^M, the negative zero of ReF collides with the physical zero of ReF', while the positive zero of ReF and the unphysical zero of ReF' coalesce at the point $z = 1$. In fig. 3, we display $|F|$ for the same three values of δ_2. Note that $|F|$ has a negative, physical zero at the critical points δ_2^m and δ_2^M. At intermediate values of δ_2, the zeros of $|F|^2$ are all outside the real interval $-1 \leqslant z \leqslant 1$, as we shall see more clearly in a moment. Finally, in fig. 4 we plot the Martin [4] parameter

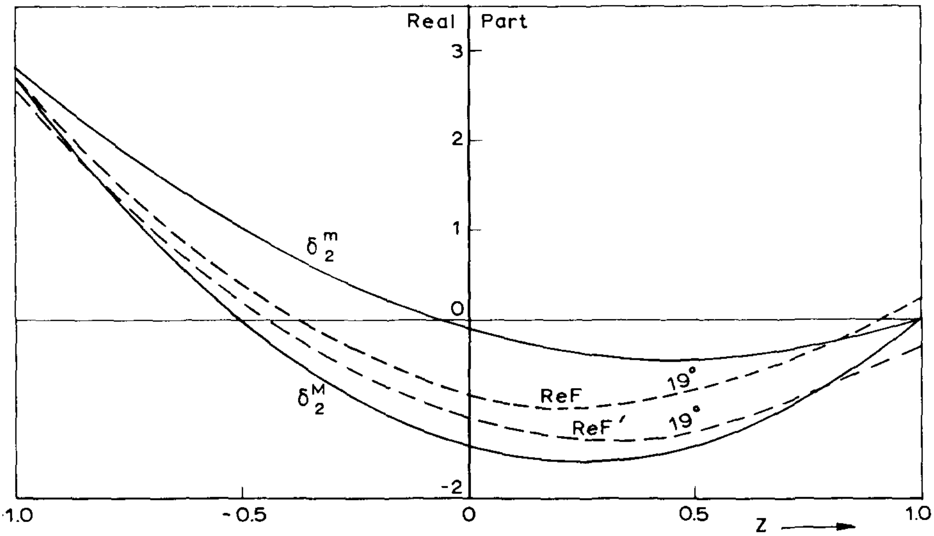

Fig. 2. The real part of the amplitude for $\delta_2 = \delta_2^m$ and $\delta_2 = \delta_2^M$ (solid curves) and for $\delta_2 = 19°$ (dotted curves).

$$\sin\mu = \max_{(1,2)} \frac{\int |F(13)|\,|F(23)|\,\mathrm{d}\Omega_3}{4\pi|F(12)|} \tag{10}$$

against δ_2. Observe that this parameter is always much greater than 0.79, which is the greatest value for which a general uniqueness theorem for the phase shifts has been proved [2,4]. The minimum value of $\sin\mu$ in fig. 4 is about 2.6 and occurs at $\delta_2 \approx 15°$, while $\sin\mu \to \infty$ at the critical points $\delta_2 \to \delta_2^m, \delta_2^M$.

The Crichton ambiguity may be examined from a geometrical viewpoint as follows: Let us write the amplitude in the form

$$F(z) = \tfrac{15}{2} f_2 (z+\alpha)(z+\beta)\,, \tag{11}$$

where α and β are certain complex numbers, and $f_2 = e^{i\delta_2}\sin\delta_2$ is the D-wave amplitude, which will be unitary if we require δ_2 to be real. The S and P waves are

$$f_0 = \tfrac{5}{2}(3\alpha\beta+1) f_2\,, \tag{12}$$

$$f_1 = \tfrac{5}{2}(\alpha+\beta) f_2\,, \tag{13}$$

and they will be unitary if we impose

$$|1 + 2if_0| = 1 = |1 + 2if_1|\,. \tag{14}$$

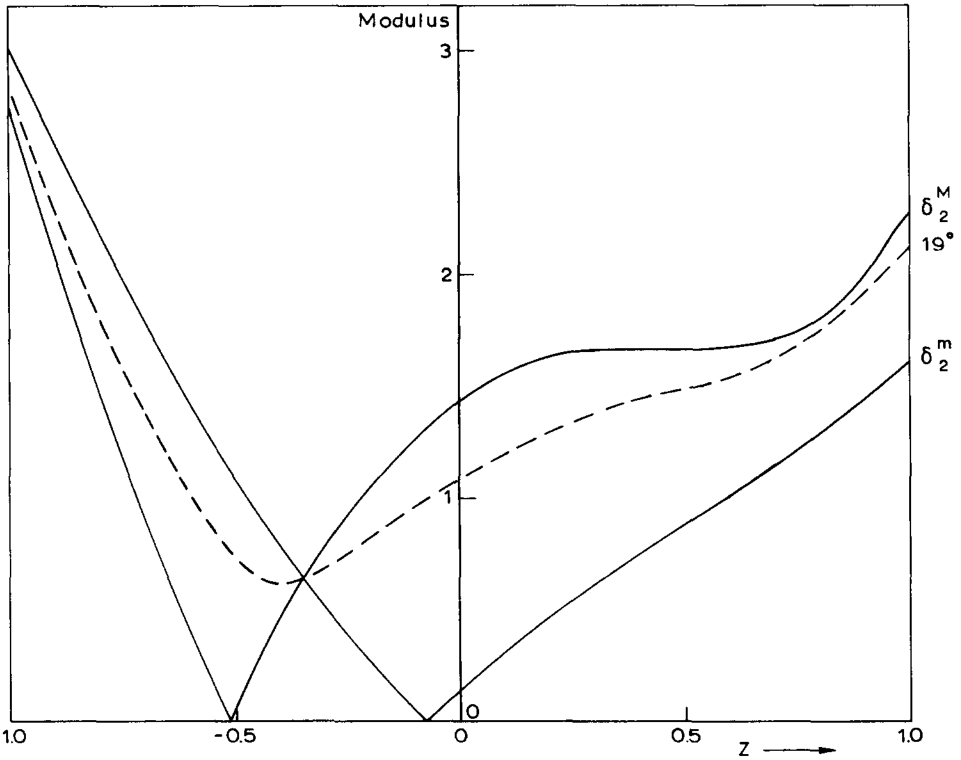

Fig.3. The modulus of the amplitude for the same δ_2-values as in fig. 2.

We now define

$$F'(z) = \tfrac{15}{2}\, f_2(z+\alpha)(z+\beta^*) \ , \tag{15}$$

so that eq. (2) is satisfied, and f_0' and f_1' are then written as in eqs. (12) and (13), but with $\beta \to \beta^*$. These waves will also be unitary if

$$|1+2if_0'| = 1 = |1+2if_1'| \ . \tag{16}$$

The four real equations (14) and (16) constrain the two complex numbers α and β. We may write eq. (14) in the form

$$|15\alpha f_2|\,\left|\beta+\frac{1}{3\alpha}-\frac{i}{15\alpha f_2}\right| = 1 \ , \tag{17}$$

$$|\,5f_2\,|\,\left|\beta+\alpha-\frac{i}{5f_2}\right| = 1 \ . \tag{18}$$

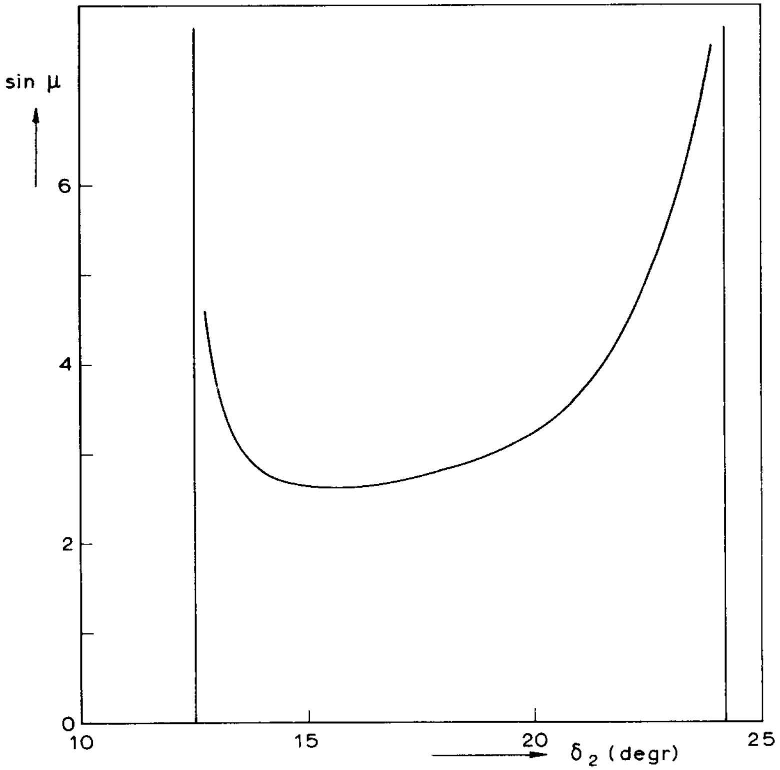

Fig. 4. The Martin parameter, sin μ, plotted against the D-wave phase shift.

For given f_2 and α, these equations define two circles in the complex β-plane. A necessary condition for the circles to intersect in two complex-conjugate points is that the centres lie on the real axis, and this will be so if

$$\alpha = -\tfrac{4}{5} + \tfrac{1}{5} i \cot \delta_2 \,. \tag{19}$$

Since we obtain F' from F by complex-conjugating β, it follows that all we have to do now is to find the two points of intersection of the circles (17) and (18), and to assign one point to β and the other to β^*. We note that eq. (11), with the value (19) for α, was obtained earlier by Martin [4].

We insert the value (19) into (17) and (18) and obtain

$$|\beta - \tfrac{1}{3}| = [9(1+15\sin^2 \delta_2)]^{-\frac{1}{2}} \overset{\text{def}}{=} r_0 \,, \tag{20}$$

$$|\beta - 1| = [5 \sin \delta_2]^{-1} \overset{\text{def}}{=} r_1 \,. \tag{21}$$

The two circles intersect if, and only if

$$|r_0 - r_1| \leqslant \tfrac{2}{3} \leqslant r_0 + r_1 \,, \tag{22}$$

and this implies

$$500p^4 - 300p^3 + 70p^2 - 20p + 3 \leqslant 0 \,, \tag{23}$$

where $p = \sin \delta_2$. This gives (for $p > 0$),

$$0.216\,950\,11 \leqslant p \leqslant 0.409\,179\,91 \,, \tag{24}$$

and these numbers are then $\sin \delta_2^m$ and $\sin \delta_2^M$. It is a straightforward matter to calculate the intersection points when $\sin \delta_2$ is in the above range, and then to compute f_0 and f_1 from eqs. (12) and (13), and similarly for f_0' and f_1'. This will of course finally yield the same answer for the phase-shifts as in the trigonometrical solution (3), (4), (7) and (8). At the critical points, the circles just touch, and then

$$\beta^{m,M} = 1 - \frac{1}{5 \sin \delta_2^{m,M}} \tag{25}$$

is a real number. As we see from eq. (11), $|F(z)|$ has a physical zero at $z = -\beta^{m,M}$ for $\delta_2 = \delta_2^{m,M}$. One finds $\beta^m = +\,0.078\,129\,07$ and $\beta^M = +\,0.422\,275\,39$. The real part may be written at the critical points as

$$\mathrm{Re}\, F(z) = \tfrac{15}{4} \sin(2\delta_2^{m,M})\, [z-1]\, [z+\beta^{m,M}] \,, \tag{26}$$

from which we see explicitly that $\mathrm{Re}F(1) = 0$. In fact, it is possible to show that, whenever there is a Crichton-like ambiguity resulting from the complex conjugation of just one zero, then $\mathrm{Re}F(1) = 0$ at a bifurcation point.

It is a pleasure to acknowledge helpful discussions with R.L. Warnock. One of the authors (M. de Roo) has carried out this work as a scientific staff member of the Stichting F.O.M. (Foundation for Fundamental Research of Matter) which is financially supported by the Netherlands Organization for Pure Scientific Research (Z.W.O.).

References

[1] J.H. Crichton, Nuovo Cimento 45A (1966) 256.
[2] D. Atkinson, P.W. Johnson and R.L. Warnock, Comm. Math. Phys. 28 (1972) 133;
D. Atkinson, G.R. Bart, P.W. Johnson and R.L. Warnock, A continuum ambiguity of phase-shift analysis in the inelastic region, submitted to the 16th Int. Conf., NAL, University of Chicago, 1972 (unpublished).
[3] D. Atkinson, G. Mahoux and F.J. Yndurâin, Nucl. Phys. B54 (1973) 263.
[4] A. Martin, Nuovo Cimento 59A (1969) 131.